# Rigorous Determination of The Time Averaging Interval to Produce A Slowly Varying Amplitude Electric Field for

# The Free Electron Laser

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Free Electron Lasers (FEL's) generate some of the world's most intense x-rays by passing a relativistic electron beam through a device of alternating magnetic fields called an undulator. The radiation from the FEL is highly collimated in the cone with aperture angle $1/\gamma$ in the forward direction. The electric field of this radiation is highly oscillatory and can be described with a slowly varying envelope or amplitude function. Appropriate averaging of the electric field must be done to remove the highly oscillatory portion and leave the envelope. The averaging is done over the interval $\frac{\lambda}{c} \ll \Delta t \ll N_u \frac{\lambda}{c}$ which allows for the fast oscillations to be averaged away but is small enough that the slowly varying envelope can be considered constant and be retained. In this paper I will determine this interval through rigorous analytical treatment to avoid the usual

## Introduction

In this paper I will give a more rigorous derivation of the Free Electron Laser (FEL) equation, given by eq. 1 below [1]. This equation describes the electric field generated by relativistic electrons traveling through an undulator by considering only a slowly varying amplitude. The electric field is usually written as in equation 2 where it is thought of as a slowly varying amplitude or envelope and a fast varying/highly oscillatory part. It is possible to solve for the envelope function $\tilde{E}(z, t)$, however the physics does not put the restriction that it will be slowly varying and so appropriate averaging must be done. It is here in this paper where I will derive the appropriate time interval $\Delta t$ that the averaging must be done over.

$$2ik\left[\left(\frac{1}{c}\frac{d}{dt}\right)+\left(\frac{d}{dz}\right)\right]\tilde{E}(z,t)=\frac{iQeK\omega_0}{c\Delta\tau}\sum_j\frac{1}{\gamma_j\beta_j}e^{i\omega_0\left(t_j(z)-\frac{z}{c}\right)}cos(k_u z) \qquad (1)$$

$$E(z,t)=\tilde{E}(z,t)e^{-i\omega_0\left(t-\frac{z}{c}\right)} \qquad (2)$$

The electric field is considered to behave in a manner similar to figure 1 below, and from this figure the idea of the slowly varying amplitude/envelope can be easily seen [1]. The electric field is a sum of the contributions from each electron, and it is from this superposition which provides fig. 1. The equation for the electric field contains both a slowly varying portion, the envelope, and the fast varying portion seen in the figure. By first removing the fast oscillations and the averaging of an appropriate Δt the envelope can be considered constant during that interval, leaving it unchanged by the averaging. This appropriate interval is determined in this paper. In this paper I will first show the usual derivation of equation 1 and then show the method with which the $\Delta t$ interval can be determined more rigorously. However let's first quickly go over some background for FEL's.

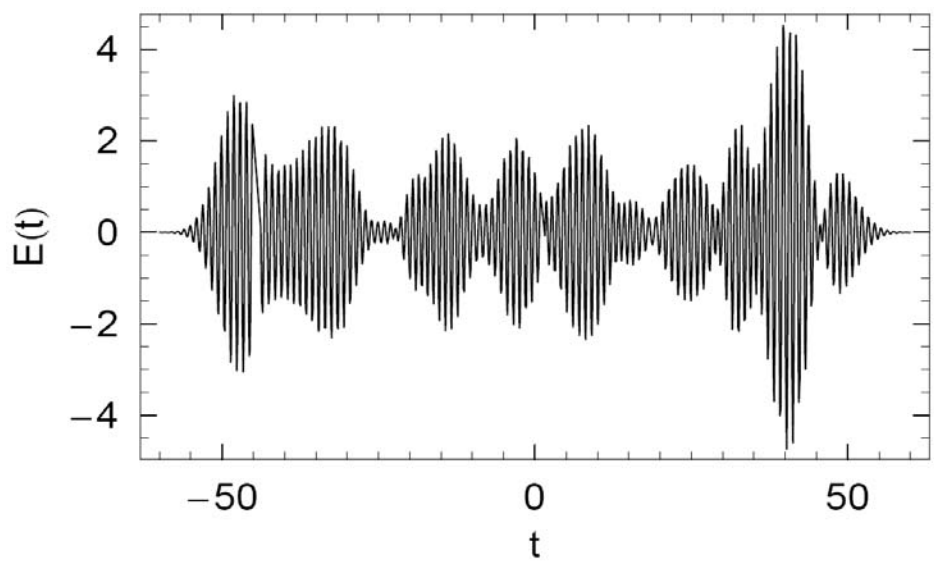


**Figure 1**: plot of the Electric Field E as a function of time[1].

# Free Electron Laser (FEL)

A free electron lasers are coherent and tunable radiation sources based on the radiation of ''free" electrons traveling in an accelerator, in comparison to common quantum lasers based on electrons bound in atomic or molecular systems which have limited tunability. Free electron lasers rely upon relativistic electrons traveling through magnetic structures called undulators which consist of alternating magnetic fields, an example is shown below in figure 2. These alternating magnetic fields force the electrons to oscillate in the transverse plane creating a sinusoidal path through the undulator, due to the Lorentz force on the electrons from the magnetic field and the acceleration of the electrons on this path causes the release of synchrotron radiation.

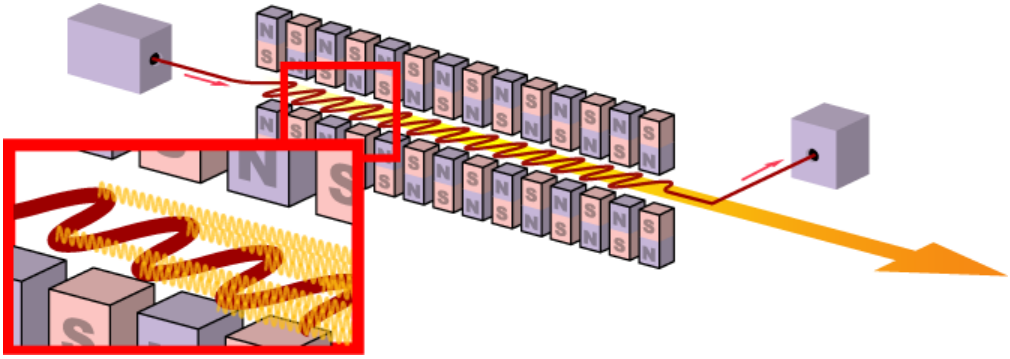

**Figure 2**: FEL undulator assembly. The yellow represents the emitted radiation.

The synchrotron radiation emitted by the electrons has many useful and remarkable features. Due to the dependence of the energy of the electrons on the wavelength of the radiation it is possible to produce radiation from the microwave, and infrared, to the visible spectrum, to ultraviolet, to X-ray wavelengths. Also, since the electron motion is in phase with the field of the radiation already emitted, the fields add together (coherently)[2]. Another feature of FEL's is Self Amplified Spontaneous Emission (SASE) which leads to the micro bunching of the electrons. Initially all electrons are evenly distributed but through the interaction of the oscillating electrons with the emitted radiation, the electrons drift into micro bunches separated by a distance equal to one wavelength of the radiation. Through this arrangement, all the radiation emitted can reinforce itself perfectly whereby wave crests and wave troughs are always superimposed on one another in the best possible way. This is what leads to the high intensities and the laser-like properties[2].

The radiation produced by Free Electron Lasers is used in many different fields ranging from Chemistry, Biology, Pharmaceutical, Materials Science, and Physics. The high intensity and small wavelength radiation together with the ability to produce bursts in the picoseconds range make it perfect for studying a vast number of phenomena from crystal formation, cell and protein structure, atomic processes and many medical applications as well. The military is also looking into FEL's in hopes of creating portable directed energy weapons.

There are many different FEL's operating around the world such as the Jefferson Laboratory Free Electron Laser, Vanderbilt Free Electron Laser Center, University of California Santa Barbara Free Electron Laser, and Duke Free Electron Laser Laboratory as just a few examples.

## The Usual Derivation

I start with equation 3 below, which is the 1-D inhomogeneous Electromagnetic wave equation for the Electric field which can be derived easily from Maxwell's equations. The right hand side of equation 3 comes from the current density inside the FEL and by neglecting space charge.

$$\left[\left(\frac{1}{c}\frac{d}{dt}\right)^2 + \left(\frac{d}{dz}\right)^2\right] E_x(z,t) = \frac{Q}{c^2}\frac{\partial J_x}{\partial t} \tag{3}$$

$$J_x(z,t) = ecK\cos(k_u z)\sum_j \frac{1}{\gamma_j \beta_j}\delta(t - t_j) \tag{4}$$

$$Q = -\frac{1}{\epsilon_0}$$

Next I Fourier transform from the time domain to the frequency domain which allows an easier solution and will in the end provide needed information. Equation 5 below is the Fourier Transform[3] of equation 3 and then using the Greens function solution method, and then the solution to equation 5 is given by equation 6 [4].

$$\left[\frac{\omega^2}{c^2} + \left(\frac{d}{dz}\right)^2\right] \bar{E}(z,\omega) = \frac{i\omega Q}{c^2}\bar{J} \tag{5}$$

$$\bar{E}(z,\omega) = \frac{Q}{2c}\left[\int_0^z dz'\, \bar{J}(z',\omega)\, e^{i\frac{\omega}{c}(z-z')} + \int_z^L dz'\, \bar{J}(z',\omega)\, e^{-i\frac{\omega}{c}(z-z')}\right] \tag{6}$$

The next step is to transform equation 6 from frequency space back to the time domain via the inverse Fourier Transform. Once the transform is done we have $E(z,t)$, the left hand side of equation 2. I then solve for the envelope function in equation 2 yielding equation 7 below.

$$\tilde{E}(z,t) = \frac{Q}{4\pi c}\left[\int_{-\infty}^{\infty} d\omega\, e^{-i(\omega-\omega_0)(t-\frac{z}{c})} \int_0^z dz'\, J(z',\omega)\, e^{-i\frac{\omega}{c}z'} + e^{i\omega_0(t-\frac{z}{c})} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t+\frac{z}{c})} \int_z^l dz'\, J(z',\omega)\, e^{i\frac{\omega}{c}(z')}\right] \tag{7}$$

From here I construct the set of differentials on the left hand side of equation 1, which I will denote in short hand by $D_+$ and plug in the Fourier transform of J, $J(z,\omega)$. This gives:

$$J(z,\omega) = ecK\cos(k_u z)\sum_j \frac{1}{\gamma_j\beta_j} e^{i\omega t_j(z)} \tag{8}$$

$$\begin{aligned} 2ikD_+\tilde{E}(z,t) = {} & \frac{QeK}{4\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega_0\left(t-\frac{z}{c}\right)} \cos(k_u z)\sum_j \frac{1}{\gamma_j\beta_j} e^{-i\omega\left(t-t_j(z)\right)} \\ & + \frac{-ieKQ\omega}{2\pi c} e^{i\omega_0\left(t-\frac{z}{c}\right)} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega\left(t+\frac{z}{c}\right)} \int_z^l dz' \cos(k_u z')\sum_j \frac{1}{\gamma_j\beta_j} e^{i\omega\left(t_j(z')+\frac{z'}{c}\right)} \\ & + \frac{-QeK}{4\pi} e^{i\omega_0\left(t-\frac{z}{c}\right)} \int_{-\infty}^{\infty} d\omega \cos(k_u z)\sum_j \frac{1}{\gamma_j\beta_j} e^{i\omega(t+t_j(z))} \end{aligned} \tag{9}$$

The first line of equation 9 represents the really interesting part of the expression. This is because in the FEL we are only concerned with the radiation in the forward direction. The radiation emitted in the forward direction should be much greater in magnitude than that emitted in the backward direction. We also only observe and use the forward emitted radiation, so that is the only part of equation 9 that is especially interesting. The second two lines represent left traveling waves; hence they are the backward radiation and will be neglected. We are therefore left with:

$$2ikD_+\tilde{E}(z,t) = \frac{iQeKk}{4\pi}\left[\int_{-\infty}^{\infty} d\omega\, e^{i\omega_0\left(t-\frac{z}{c}\right)} cos(k_u z) \sum_j \frac{1}{\gamma_j\beta_j} e^{-i\omega\left(t-t_j(z)\right)}\right]$$
$$= \frac{iQeK\omega_0}{c} e^{i\omega_0\left(t-\frac{z}{c}\right)} cos(k_u z) \sum_j \frac{1}{\gamma_j\beta_j}\delta(t-t_j(z)) \qquad (10)$$

At this point we have an expression for what I have said is the envelope function of the electric field from equation 2. However nothing in the physics or the derivation to this point forces the electric field envelope $\tilde{E}(z,t)$ to be slowly varying or even smooth. Now the task is to average this expression appropriately to remove any fast oscillatory pieces and to make it smooth. Because as was seen in figure 1, our envelope should vary slowly in time and be smooth. The averaging that I shall do is referred to as a delta-slice average in time. The average is performed for $\Delta t$ with, $\frac{\lambda}{c} \ll \Delta t \ll N_u \frac{\lambda}{c}$ , and at a constant value of z and by performing this averaging which I show below, we are led to equation 12.

$$\frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} 2ikD_+\tilde{E}(z,t) = \frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \frac{iQeK\omega}{c} e^{i\omega_0\left(t-\frac{z}{c}\right)} cos(k_u z) \sum_j \frac{1}{\gamma_j\beta_j}\delta(t-t_j(z)) \qquad (11)$$

$$2ikD_+\tilde{E}(z,t) = \frac{iQeK\omega_0}{c\Delta t}\sum_{j\in[\Delta t]} \frac{1}{\gamma_j\beta_j} e^{i\omega_0\left(t_j(z)-\frac{z}{c}\right)} cos(k_u z) \qquad (12)$$

This is the same as equation 1 above, the FEL equation for the slowly varying envelope of the electric field. Some important features of this expression are the right hand side is smooth, as opposed to equation 10 which contains delta functions. Equation 12 is dependent upon the interval $\Delta t$ and varies in a way that is consistent with the envelope seen in figure 1. Also, as we see in equation 2 for the current density J, the sum is over all particles producing the radiation, but now the sum is over only those particles that are in the delta-slice.

## Determination of Averaging Interval

The size of the $\Delta t$ is given above as $\frac{\lambda}{c} \ll \Delta t \ll N_u \frac{\lambda}{c}$ and was determine simply by 'physical' arguments rather than found in a mathematically rigorous fashion. It is the goal of this section to determine that interval directly and use it to obtain equation 12 above, so that we are left with an expression that experiment has confirmed is very accurate but in a way that puts the equation and its derivation on a more mathematical platform. To begin this section let's start by considering equation 6 and then construct a differential equation with the Fourier transform of the differential $D_+$ giving:

$$\left[\frac{i\omega}{c} + \frac{d}{dz}\right] \bar{E}(z,\omega) = \frac{iQ\omega}{c^2}\left[\int_0^z dz'\, \bar{J}(z',\omega)\, e^{i\frac{\omega}{c}(z-z')}\right] \qquad (13)$$

The next step in this derivation is to use the sinc function defined below [5]. This function has the property that it acts as a low pass filter, which means it filters out high frequencies leaving only low frequencies behind. We want something like this because the high frequencies are what give rise to the fast oscillations that we do not want and the lower frequencies help form the slowly varying envelope that we do want. The sinc function is shown below in figure 2, for values of x above ~20 the sinc function becomes very small (on the order of .05 or less) and tends to zero at infinity.

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x} \qquad (14)$$

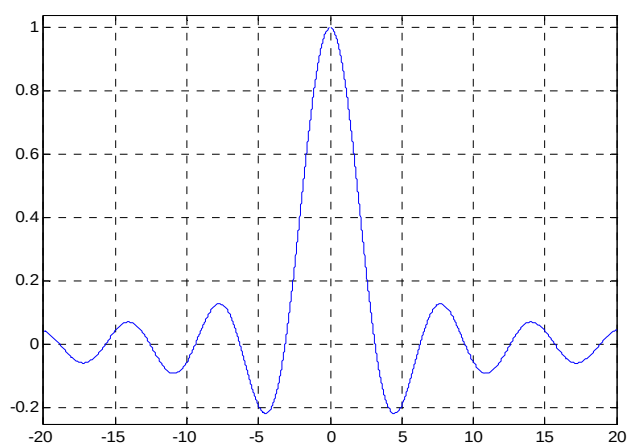


**Figure 3**: plot of sinc(x)

Instead of x however, we will use the choice of $x = \frac{\Delta t}{2}\omega$ , where $\omega$ is the frequency and $\Delta t$ is the same as above and what we are trying to place bounds on to do our averaging. Now multiply both sides of equation 13 by the sinc function giving:

$$\operatorname{sinc}\left(\frac{\Delta t}{2}\omega\right)\bar{D}_+\bar{E}(z,\omega) = \operatorname{sinc}\left(\frac{\Delta t}{2}\omega\right)\frac{iQ\omega}{c^2}\left[\int_0^z dz'\, J(z',\omega)\, e^{i\frac{\omega}{c}(z-z')}\right] \quad (15)$$

Next I insert the value of the Fourier Transform of the current density and make the assumption that $t_f = \frac{z}{c\beta}$ which is a 'prescribed' or ideal trajectory for an ideal electron passing through the undulator. By ideal, I mean that it does not interact with the emitted radiation and follows a perfect path through the undulator that is completely describable. This is done to simplify and allow for analytical treatment. Since deviations off of the true/ideal trajectory are small and the interval we are determining is not exact, meaning not an exact equality, then it will approximately cover the non ideal cases too. After these three things we are left with:

$$\hat{E}(z,\omega) = \operatorname{sinc}\left(\frac{\Delta t}{2}\omega\right)\frac{QeK}{2}\sum_f \frac{1}{\gamma_f\beta_f}\left[\frac{-i}{\left(k_u + \frac{\omega}{c}\left(\frac{1-\beta}{\beta}\right)\right)}\left(e^{i\left(k_u+\frac{\omega}{c}\left(\frac{1-\beta}{\beta}\right)\right)z} - 1\right)\right.$$
$$\left. + \frac{-i}{\left(-k_u + \frac{\omega}{c}\left(\frac{1-\beta}{\beta}\right)\right)}\left(e^{i\left(-k_u+\frac{\omega}{c}\left(\frac{1-\beta}{\beta}\right)\right)z} - 1\right)\right]$$

$$\hat{E}(z,\omega') = \operatorname{sinc}\left(\frac{\Delta t}{2}\omega'\right)\frac{QeK}{2}\sum_f \frac{1}{\gamma_f\beta_f}e^{i\frac{\omega z}{c}}\left[e^{i\left(\frac{1-\beta}{2c\beta}\right)(\omega'+2\omega_o)z}\operatorname{sinc}\left(\left(\frac{1-\beta}{2c\beta}\right)(\omega'+\omega_o)z\right)\right.$$
$$\left. + e^{i\left(\frac{1-\beta}{2c\beta}\right)(\omega')z}\operatorname{sinc}\left(\left(\frac{1-\beta}{2c\beta}\right)\omega' z\right)\right] \quad (16)$$

Where I have defined $\hat{E}(z,\omega) = \operatorname{sinc}\left(\frac{\Delta t}{2}\omega\right)\bar{D}_+\bar{E}(z,\omega)$. Looking at equation 16 above, we can see that the second term is by far the dominate term when $\omega = \omega_o$, and with the substitution that

$$k_u = \frac{\omega_o}{c}\left(\frac{1-\beta}{\beta}\right)$$

Then ignoring the first term we can rewrite the second, to get:

$$\hat{E}(z,\omega) = \mathrm{sinc}\left(\frac{\Delta t}{2}\omega\right)\frac{iQeK\omega}{2c}\sum_j \frac{1}{\gamma_j\beta_j}\left[e^{i\left(\frac{1-\beta}{2c\beta}\right)(\omega-\omega_o)z}e^{i\frac{\omega z}{c}}\mathrm{sinc}\left(\left(\frac{1}{2c\beta}\right)(\omega-\omega_o)z\right)\right] \qquad (17)$$

Next, Let

$$\omega' = \omega - \omega_o \qquad \text{and} \qquad z = L$$

Where I have chosen z=L because while we are interested in what occurs inside the undulator, our final measurable result occurs at the end of it at z=L. We are left with:

$$\hat{E}(L,\omega') = \mathrm{sinc}\left(\frac{\Delta t}{2}\omega'\right)\frac{iQeK\omega}{2c}\sum_j \frac{1}{\gamma_j\beta_j}\left[e^{i\left(\frac{1-\beta}{2c\beta}\right)\omega' L}e^{i\frac{\omega' L}{c}}\mathrm{sinc}\left(\left(\frac{1-\beta}{2c\beta}\right)\omega' L\right)\right] \qquad (18)$$

And we will now also consider the sinc function centered around the same place as the rest of the function, $\omega_o$, since everything is a function of $\omega'$. Now we make several further simplifications, such as:

$$L = N_u\lambda_u$$

$$\lambda = \frac{\lambda_u}{\beta}(1-\beta)$$

And we are then left with:

$$\hat{E}(L,\omega') = \mathrm{sinc}\left(\frac{\Delta t}{2}\omega'\right)\frac{iQeK\omega}{2c}\sum_j \frac{1}{\gamma_j\beta_j}\left[e^{i\frac{N_u\lambda}{2c}\omega'}e^{iN_u 4\pi\gamma^2}\mathrm{sinc}\left(\frac{N_u\lambda}{2c}\omega'\right)\right] \qquad (19)$$

Now, the first two terms inside the brackets only affect the phase of the sinc function and does not affect the magnitude of the entire function. Therefore we have that the overall behavior is that of the sinc function inside of the brackets. The goal now is to determine bounds on $\Delta t$ which we will do by considering the behavior of the sinc function and what type of information we want to keep from this expression. Looking back at figure 2 we see that the sinc function has

some characteristic width that is inversely proportional to the constant multiplying $\omega'$ (this in our case is $\frac{\Delta t}{2}$). Therefore decreasing $\Delta t$ widens the filtering sinc function.

We want the characteristic width of the sinc function to be large enough to encompass the interesting features of the electric field but small enough to filter out the high frequency carrier oscillation $\sim\omega_o$ . The latter constraint implies that

$$\frac{\lambda}{c} = \frac{1}{\omega_o} \ll \Delta t$$

While the former constraint can be imposed by requiring the width of the filtering function to be much greater than the characteristic width of the spontaneous radiation. From Equation 19, we deduce that the spontaneous radiation has a bandwidth of order $\frac{c}{N_u \lambda}$. Requiring the filter width to encompass this frequency range implies that

$$\Delta t \ll \frac{N_u \lambda}{c}$$

Therefore by using the sinc function as a filter was are left with the appropriate averaging interval given now by equation 20, over with we are able to find the slowly varying amplitude electric field.

$$\left[\!\left[\frac{\lambda}{c} \ll \Delta t \ll \frac{N_u \lambda}{c}\right]\!\right] \qquad (20)$$

Lastly I would like to look at how the use of the sinc function is a natural choice for this problem besides its use as a low pass filter. We begin by considering equation 10 in a very general form:

$$D_+ \tilde{E}(z,t) = f(z,t) \qquad (21)$$

Where in this case the right hand side of equation 21, $\bar{f}(z,\omega)$, represents the right hand side of equation 6, so that

$$f(z,t) = \frac{iQeB\omega_0}{c} e^{i\omega_0\left(t-\frac{z}{c}\right)} \cos(k_u z) \sum_j \frac{1}{\gamma_j \beta_j} \delta(t \quad t_j(z))$$

Then if we apply the delta slice average to equation 21 we have basically equation 11:

$$\frac{1}{\Delta t} \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} dt' D_+ \tilde{E}(z,t') = \frac{1}{\Delta t} \int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} dt' f(z,t')$$

But this can be rewritten using step functions as,

$$\int_{-\infty}^{\infty} dt' D_+ \tilde{E}(z,t') G(t-t') = \int_{-\infty}^{\infty} dt' f(z,t') G(t-t') \qquad (22)$$

$$G(t-t') = \frac{1}{\Delta t}\left[\theta\left(\frac{\Delta t}{2} - (t-t')\right) - \theta\left(-\left(\frac{\Delta t}{2} + (t-t')\right)\right)\right]$$

But now this is in the form of a convolution on both sides and if we represent the convolution by * we have equation 22 looking like:

$$D_+ \tilde{E}(z,t) * G(t) = f(z,t) * G(t)$$

And if we Fourier transform this equation, because of the properties of the transform we will have the transform of each term multiplied instead of convoluted, leaving

$$\bar{D}_+ \bar{E}(z,\omega) \bar{G}(\omega) = \bar{f}(z,\omega) \bar{G}(\omega) \qquad (23)$$

But the interesting piece is $\bar{G}(\omega)$ and so let's look at it more closely.

$$G(t) = \frac{1}{\Delta t}\left[\theta\left(\frac{\Delta t}{2} - (t)\right) - \theta\left(-\left(\frac{\Delta t}{2} + (t)\right)\right)\right]$$

Which after Fourier transforming it leaves us with our choice of the sinc function, and now puts equation 23 is in the same form as equation 15 with

$$\bar{G}(\omega) = \operatorname{sinc}\left(\frac{\Delta t}{2}\omega\right)$$

$$\operatorname{sinc}\left(\frac{\Delta t}{2}\omega\right)\bar{D}_{+}\bar{E}(z,\omega) = \operatorname{sinc}\left(\frac{\Delta t}{2}\omega\right)\bar{F}(z,\omega)$$

Therefore our choice in using the sinc function that we did was a natural one since it is equivalent to the delta slice in the time domain via the Fourier Transforms.

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